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CONDITIONALIZATION AND OBSERVATION*

0. INTRODUCTION

I take bayesianism to be the doctrine which maintains that (i) a set of reasonable beliefs can be represented by a probability function defined over sentences or propositions, and that (ii) reasonable changes of belief can be represented by a process called conditionalization. Bayesians have produced several ingenious arguments in support of (i); but the equally important second condition they often seem to take completely for granted. My main aim is to fill this gap in those bayesian positions which characterize reasonable belief directly as a probability function. Thus, what follows applies equally to the bayesian personalists' views which characterize reasonable belief as having subjective sources and to views such as that of Carnap which attempt to explicitly define a function which characterizes the degree of belief it would be objectively reasonable for an idealized rational agent to have in a given proposition in stated circumstances. Many frequentist views are also classifiable as bayesian, and I will briefly discuss the justification of condition (ii) from the point of view of a frequency interpretation of probability or reasonable degree of belief. Along the way I will also have occasion to touch on the connection between conditionalization and observation.

Throughout the discussion I will rely on several common bayesian presuppositions. The object of study is a notion of belief, perhaps most aptly described as degree of confidence, which can be ordered as to strength and admits of quantatization. Such beliefs, or degrees of confidence, are assumed to reveal themselves in an agent's disposition to make bets voluntarily or under duress. The agent whose beliefs are under discussion is assumed to be an ideal logician, and the set of propositions about which the agent has beliefs is assumed to be closed under all logical operations. Also, the set of propositions for which the agent entertains beliefs is assumed to be fixed. Quite clearly, when this assumption is violated, the bayesian model does not apply; and the most cogent arguments against

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conditionalization seem to turn on cases in which this assumption clearly fails (see, e.g., [10] *passim*). Finally, it is assumed that no such set of beliefs, taken together, are reasonable unless they can be described by a function satisfying the axioms of probability, where the numbers given by such a function can be taken to represent what the agent regards as fair betting quotients for those propositions on which it makes sense to bet. Clearly these assumptions are idealizations, and conclusions which depend on them can be applied to real cases only insofar as the idealizations are in relevant respects sufficiently close approximations to correct descriptions.

When an agent changes his beliefs his new beliefs may fail to be reasonable as a result of a number of different factors. For example, there may be nothing wrong in the way he changes his beliefs, but the old beliefs may have been unreasonable, and the new beliefs may inherit this irrationality. On the other hand, the old beliefs may have been perfectly reasonable, while the fault lies entirely with the pattern of change of belief. Preanalytically, there is a distinction to be made between the rationality of beliefs at a time and the rationality of the method or pattern of change of belief, where the latter figures in but does not by itself determine the former. The relation is complex, as is illustrated by the example of an agent who starts with unreasonable beliefs and changes them by a method we would all applaud if his old beliefs had been reasonable. In such a case the new beliefs will still be correctly described as reasonable if the agent had no way of telling that his original beliefs were unreasonable, if he had every reasonable belief that, counter to fact, his original beliefs were perfectly reasonable. In view of this sort of complication a precise account of the distinction between reasonable belief at a time and reasonable *change* of belief would require a precise way of treating an agent's second order beliefs about his beliefs and the second order rationality of the agent's belief about the rationality of his beliefs; and these tools are not yet available. However, the following rough and ready definition should make the distinction sufficiently clear for use in what is to follow:

(0.1.1.) A *change of belief is reasonable* if and only if
(a) the new beliefs are reasonable,

or

(b) both the new and the old beliefs are not reasonable, but the new beliefs would have been reasonable if (i) the old beliefs had been reasonable and (ii) both before and after the change the agent has a high reasonable degree of belief that his old degrees of belief were reasonable.

A change from one set of beliefs to another will be said to be described by conditionalization on the proposition E if the old set of beliefs is described by the probability function P_o, and for every proposition A, in the domain of P_o, the new degree of belief in A is given by $P_n(A) = P_o(AE)/P_o(E)$. I will use $P(A/E)$ as a notational variant for $P(AE)/P(E)$, and throughout I will use 'is determined by', 'arises by', 'is given by' and 'takes place by' interchangeably with 'is described by', as stylistic variants in expressions such as 'change of belief is described by conditionalization'.

The problem of justifying conditionalization may now be loosely stated as the problem of supporting the claim that changes of belief described by conditionalization on some proposition which the agent has learned to be true (for example, by observation) are reasonable changes. More precisely, one might try to argue that all and only changes of belief by conditionalization are reasonable. Or one might try to show that all changes of belief by conditionalization are reasonable, while leaving it open whether or not other changes of belief might also be reasonable. Both of these options seem wholly unrealistic. Finally, one might argue that, under certain well specified conditions, only changes of belief by conditionalization are reasonable, that is that if any change of belief is reasonable, then such reasonable change of belief must be described by conditionalization on some proposition E. Most of the arguments which follow will be of this third form.

Finally it is to be noted that when change of belief is described by conditionalization on a proposition, E, the new degree of belief in E is 1, and it is often supposed that $P(E) = 1$ when the agent *knows* that E is true. I regard ascription of degree of belief of 1, and the associated concepts of certainty and knowledge, as at best idealizations; and fortunately there is a generalization of conditionalization which requires none of them. However, in Part 1 I will assume that agents do gain knowledge which makes degree of belief of 1 reasonable, and that only knowledge makes degree of belief of 1 reasonable. I make these assumptions here because I wish to present the arguments in a form suitable for those who accept them and because the arguments of Part II, which do not require these assumptions, are easiest to present as straightforward generalizations of the arguments given in Part I.

PART I

1.1. *Introduction*

There are four well-known types of arguments for the probability axioms: (1) Arguments from frequency definitions of probability; (2) 'Dutch Book' arguments which show that if an agent's belief function violates the probability axioms, his betting rates make him susceptible to forming collections of bets on which he will lose no matter what happens (e.g., [3], pp. 102–4); or for which no matter what happens he may lose but cannot gain (e.g. [9]); (3) Arguments from qualitative assumptions about belief and preference (e.g., [8], pp. 6–43); and (4) Arguments from assumptions about certain continuity properties of an agent's belief function (e.g. [2], [4]). In this part I will discuss arguments for conditionalization which are analogues of the first three types of arguments for conditionalization. (At the present time I have not investigated the possibility of analogues of the fourth kind of argument.) Sections 1.2. and 1.3. will very briefly examine the frequency and Dutch Book type of arguments. Sections 1.4 to 1.6. will develop in detail a pattern of argument which rests on a qualitative assumption about inductive reasoning.

1.2. *Frequency Arguments*

If probability or reasonable degree of belief is defined in terms of relative frequency, the claim that new probabilities or reasonable degrees of belief are given by conditionalization on the old ones follows trivially from the definitions, though details differ with differing frequentist analyses. I will here give only a brief sketch of how such arguments proceed. Note that in this section capital letters are used as parameters for properties instead of, as elsewhere, propositions.

On a frequentist view the probability or reasonable degree of belief that a case has property A is defined as the frequency (observed or 'limiting') of cases that have property A among some non-empty reference class of cases which have property R. This will be written

$$P_R(A) =_{\text{def}} F(AR/R).$$

Suppose that the agent comes to know that (and no more than that) the case in question has property B as well as property R. Then, on any reasonable analysis of 'reference class' the reference class becomes the class of all cases which have both property R and property B. Thus the new probability or reasonable degree of belief is

$$(1.2.1.)\quad P_{RB}(A) = F(ARB/RB).$$

If the function F is being interpreted as a finite frequency we note that the reference class was assumed non-empty, and since the agent knows the case in question has property B, the class of cases which have both properties B and R, is non-empty. If F is interpreted as a limiting frequency we need to assume that $F(BR/R) \neq 0$. Using this assumption if needed it follows from (1.2.1.) that

$$\begin{aligned} P_{RB}(A) &= F(ARB/R)/F(BR/R),\\ &= P_R(AB)/P_R(B). \end{aligned}$$

Consequently, the new probability for the case having property A is given by conditionalization from the old, provided that the observed property B did not initially have probability zero, interpreted as a limiting relative frequency.

1.3. *The Dutch Book Argument*

In this section I report a version of the Dutch Book Argument devised by David Lewis.[1]

Let P_o and P_n be, respectively, the agent's old belief function at time o and his new belief function at time n. For this argument we must assume that the agent's belief function, P, represents his betting rates, so that, for any proposition A, in the domain of P, $P(A)$ is the price for which he would be indifferent between buying or selling the bet $\begin{bmatrix} 1 \text{ if } A \\ 0 \text{ otherwise} \end{bmatrix}$; and we must assume that the domain of P_o and P_n includes a set $\{E_i\}_{i \in I}$ of mutually exclusive and jointly exhaustive propositions that specify, in full detail, all the alternative courses of experience the agent might undergo between time o and time n. Let $\{E_i\}_{i \in I'}$ be the subset of $\{E_i\}_{i \in I}$ such that $P_o(E_i) > 0$ for $i \in I'$. A version of the Dutch Book Argument due to Abner Shimony [9] supports the claim that $I = I'$. If $I = I'$ is not assumed, the following argument supports change of belief by conditionalization for cases in which some E_i, $i \in I'$ occurs, while giving no information for cases in which some E_i, $i \notin I'$ occurs. The argument shows that if at time o the agent knows, for some $i \in I'$ what his new belief function, P_n will be if E_i should turn out to be true, and if for such an $i \in I'$ the function to be adopted, if E_i turns out to be true, is not $P_n(A) = P_o(A/E_i) =_{\text{def}} P_o(AE_i)/P_o(E_i)$, then a bookie who knows no more nor less than the agent can induce the agent to buy and sell bets on which he will have a net loss whatever happens. If one agrees that any plan for changing belief is unreasonable if it makes one vulnerable in this way to certain loss, the conclusion can be summarized thus: No explicitly formulated plan for changing beliefs in the face of new evidence is reasonable unless, for any $i \in I'$ for which the plan specifies the beliefs to be adopted should E_i occur, the plan calls for changing beliefs by conditionalization on E_i if E_i occurs.

The bookie's system for exploitation of the non-conditionalizing agent proceeds as follows: Suppose, for some $i \in I'$ and A, the agent plans to have new beliefs $P_n(A) < P_o(A/E_i)$ if E_i turns out to be true. Let $x = P_o(A/E_i)$ and $y = P_o(A/E_i) - P_n(A)$.

At time o the bookie sells the agent the bets

(a) $\begin{bmatrix} 1 & \text{if} & AE_i \\ 0 & \text{otherwise} \end{bmatrix}$

(b) $\begin{bmatrix} x & \text{if} & \bar{E}_i \\ 0 & \text{otherwise} \end{bmatrix}$

and

(c) $\begin{bmatrix} y & \text{if} & E_i \\ 0 & \text{otherwise} \end{bmatrix}$

for the maximum price he will pay, namely $P_o(AE_i) + xP_o(\bar{E}_i) + yP_o(E_i) = P_o(A/E_i) + yP_o(E_i)$. If E_i turns out to be false the agent wins $P_o(A/E_i)$ on bet (b) and has a net loss of $yP_o(E_i)$. If E_i turns out to be true the bookie buys back the bet $\begin{bmatrix} 1 \text{ if } A \\ 0 \text{ otherwise} \end{bmatrix}$ for the minimum price the agent will pay. By hypothesis this is $P_n(A) = P_o(A/E_i) - y$. The agent wins y on bet (c). His total gain is $P_o(A/E_i) - y + y$, his total loss is $P_o(A/E_i) + yP_o(E_i)$, and he again has a net loss of $yP_o(E_i)$. Since, by assumption, $P_o(E_i) > 0$ and $y > 0$, the agent loses the positive amount $yP_o(E_i)$ whatever happens. If the agent plans new beliefs $P_n(A) > P_o(A/E_i)$ instead of $P_n(A) < P_o(A/E_i)$

the argument proceeds in exactly the same way except that the bookie buys (sells) where in the foregoing argument he sells (buys).

The argument can be rephrased by noting that bets (a) and (b), taken together, can be viewed as a 'conditional bet' on which no one has net gain or loss if E_i turns out to be false and which becomes a bet on A if E_i turns out to be true. The bookie sells the agent this conditional bet and also (c), a bet at small stakes on E_i. If E_i turns out to be false the agent neither gains nor loses on the conditional bet, but loses on bet (c). If E_i turns out to be true, the bookie buys back the (now no longer conditional) bet on A at a reduced rate. The agent wins on bet (c), but the bookie has been careful to set the stakes small enough on (c) so that those winnings do not offset the agent's loss.

The bets (a) and (b) are just the ones used in the well-known application of the Dutch Book Argument which shows that to avoid vulnerability to certain loss one must use $P_o(AE_i)/P_o(E_i)$ as one's betting rate on the bet conditional on E_i, to be called off if E_i is false and to become a bet on A if E_i is true. Previous authors (e.g. Hacking [5], p. 315) concluded that this fact about conditional betting rates could not be applied in a Dutch Book type of argument to reach conclusions about change of belief. They were mistaken because they failed to consider application of the argument to a set of propositions meeting the special conditions specified above for $\{E_i\}_{i\in I'}$. Indeed it is easy to see that the above pattern of argument applies *only* to the propositions of $\{E_i\}_{i\in I'}$. Suppose a proposition F is incompatible with all the members of $\{E_i\}_{i\in I'}$. Then, by hypothesis $P_o(F)=0$, and the fractions $P_o(AF)/P_o(F)$ needed for the argument don't exist. Suppose F implies E_i, for some $i\in I'$, but E_i does not imply F. If the bookie is to exploit the agent by making the relevant bets on F, he must be able to determine whether or not F is true. But since the E_i, $i\in I$ are assumed to describe, in full detail, the various courses of experience which the agent might undergo between time o and time n, the agent learns only whether E_i is true while the bookie learns whether F is true. Exploitation by dint of such greater knowledge or keener powers of observation shows nothing derogatory about the agent's plan for change of belief. Suppose F is compatible with two or more members of $\{E_i\}_{i\in I'}$, but is not merely the disjunction of two or more members of $\{E_i\}_{i\in I'}$. Then there is at least one possible outcome in which the bookie will not be able to determine whether F is true merely by knowing which member of $\{E_i\}_{i\in I}$ is true.

Consequently the bookie cannot *whatever happens* take advantage of the agent without knowing more or having keener powers of observation. As the final possibility, F might be a disjunction, $F = \bigvee_{j \in J} E_j$, where $J \subset I'$. Then, either $P_o(A/E_j) = P_o(A/F)$ for all $j \in J$, in which case the argument applied to F yields no different results than when applied to members of $\{E_i\}_{i \in I'}$. Or, for some k, $l \in J$, $P_o(A/E_k) < P_o(A/F) < P_o(A/E_l)$. In this case, the bookie cannot inflict sure loss because he cannot tell for sure whether he should buy or sell the initial bets (a)–(c).

1.4. *The Equivalence of Conditionalization and a Qualitative Condition on Change of Belief*

I turn now to characterizing conditionalization in terms of a qualitative equivalent. One way of pursuing this tack is to draw a connection between conditionalization and the qualitative interdependencies between belief, preference, and change of preference as developed by Savage [8]. This approach results in some interesting connections between change of belief and change of preference; and it suggests an argument for conditionalization which, however turns out to beg the question at issue.[2]

I will here proceed instead by detailing and then applying a very simple qualitative characterization of conditionalization.

I will throughout suppose that a change of belief takes place, that P_o describes the agent's beliefs before the change, and that P_n describes his beliefs after the change. I will also throughout suppose that propositions range over just those propositions which are in the agent's domain of beliefs. I will say that proposition E has condition C, or $C(E)$ for short, just in case the agent's belief in E changes from something greater than zero to unity, and furthermore, for any two propositions, A and B, each of which logically implies E, if A and B are believed equally before the change, then they are believed equally after the change. If we use '$\Rightarrow$' to mean 'logically implies', then $C(E)$ is defined precisely by

$$(1.4.1.)\quad C(E) \equiv_{\text{def}} 0 < P_o(E) < 1 \;\&\; P_n(E) = 1 \;\& \\ (\forall A)(\forall B)[(A \Rightarrow E \;\&\; B \Rightarrow E \;\&\; P_o(A) = P_o(B)) \rightarrow \\ P_n(A) = P_n(B)].$$

Note that although this condition has been stated in terms of the equality of two quantitative degrees of belief, no more is really required than that if certain of the agent's beliefs are qualitatively equal before the change, that is if neither is stronger than the other, then they are qualitatively equal at the conclusion of the change. This qualitative condition on change of belief, together with the assumption we are making throughout, that at a fixed time the agent's beliefs can be represented by a probability measure, will suffice for what follows.

I will use 'Cond(E)' to state that change of belief takes place by conditionalization on E; more exactly,

(1.4.2.) $\mathrm{Cond}(E) \equiv_{\mathrm{def}} 0 < P_o(E) < 1 \,\&\, (\forall A)[P_n(A) = P_o(A/E)]$.

I will prove that, in the presence of certain further assumptions, $C(E)$ is equivalent to Cond(E), for any E.

First I will show that

(1.4.3.) For all E, if Cond(E), then $C(E)$.

For an arbitrary E, assume Cond(E). By definition of 'Cond' $0 < P_o(E) < 1$ and $(\forall A)(P_n(A) = P_o(A/E))$. In particular, $P_n(E) = P_o(E/E) = 1$. And for any A, B, each of which implies E,

$$P_n(A) = P_o(A/E) =_{\mathrm{def}} P_o(AE)/P_o(E) = P_o(A)/P_o(E).$$
$$P_n(B) = P_o(B/E) =_{\mathrm{def}} P_o(BE)/P_o(E) = P_o(B)/P_o(E).$$

So if $P_o(A) = P_o(B)$, it follows that $P_n(A) = P_n(B)$.

To prove the converse, I first prove the lemma that for all E,

(1.4.4.) If $0 < P_o(E) < 1$ and $P_n(E) = 1$ and $(\forall A)(P_o(A/E) = = P_n(A/E))$, then Cond$(E)$.

Let A be an arbitrary proposition; and assume the antecedent of (1.4.4.).

$$\begin{aligned} P_o(A/E) &= P_n(A/E), \\ &=_{\mathrm{def}} P_n(AE)/P_n(E), \\ &= P_n(AE) \quad \text{(because } P_n(E) = 1 \text{ is assumed)}, \\ &= P_n(A) - P_n(A\bar{E}), \\ &= P_n(A) \quad \text{(again because } P_n(E) = 1 \text{ is assumed)}. \end{aligned}$$

To finish proving the equivalence of $C(E)$ and Cond(E) we need to prove that the antecedent of (1.4.4.) follows from $C(E)$; and, as can be demonstrated by simple counterexamples, this can be done only in the presence of some further assumption. I will proceed by giving, in outline only, a very simple proof, which uses a strong further assumption. I will then provide a detailed proof which uses a weaker assumption. The first proof is mentioned because of its greater mathematical elegance and because it will enable those with background in mathematics to see very quickly what is 'really going on' in the second proof. The second proof is presented because it establishes the conclusion under an assumption weak enough to make the result useful to the bayesian characterization of treatment of change of belief.[3]

The first proof requires the definition

(1.4.5.) The agent's domain of beliefs will be said to be *full* if and only if for every number q and every proposition A in the domain such that $P_o(A) = r$ and $0 \leqslant q \leqslant r$, there is a proposition B, such that $B \Rightarrow A$ and $P_o(B) = q$.

Recall that the agent's domain of beliefs is assumed throughout to be closed under all logical operations and so forms a Boolean field. It is now easy to prove that for any E,

(1.4.6.) If the agent's domain of beliefs is full and if $C(E)$, then $(\forall A)(P_o(A/E) = P_n(A/E))$.

Let E be given. Assume $C(E)$ and that the agent's domain of beliefs is full. In the presence of the fullness assumption the last conjunct of $C(E)$ is equivalent to the existence of a function, g, defined on $[0, P_o(E)]$ such that if $A \Rightarrow E$, then $g[P_o(A)] = P_n(A)$. Since P_o and P_n are probability measures, g is a positive function and is additive, i.e., for positive x, y such that $0 < x + y < P_o(E)$, $g(x+y) = g(x) + g(y)$. It follows, though not completely trivially, that there is a constant, k, such that for all arguments, x, $g(x) = kx$. (The proof is a slight modification, devised by Arthur Fine, of the proof of Theorem 1, p. 34, in Aczel [1].) So for arbitrary A, and using the first two conjuncts of $C(E)$ which give $P_o(E) \neq 0$ and $P_n(E) \neq 0$, we have

$$\begin{aligned} P_n(A/E) &= {}_{\text{def}} P_n(AE)/P_n(E), \\ &= g[P_o(AE)]/g[P_o(E)], \quad \text{since } AE, E \text{ each imply } E, \\ &= kP_o(AE)/kP_o(E), \\ &= P_o(AE)/P_o(E), \\ &= {}_{\text{def}} P_o(A/E). \end{aligned}$$

The second proof will use the definition

(1.4.7.) The agents domain of beliefs is *full enough at E* if and only if

(i) If $B \Rightarrow E$ and $P_o(B) = (r/s)P_o(E)$, for integers r, s, then there is a sequence of propositions $\{X_i\}_{i=1}^{s}$ such that, for $1 < i < j < s$,

(ia) X_iX_j is logically false,

(ib) $\bigvee_{i=1}^{s} X_i$ is logically equivalent to E,

and

(ic) $P_o(X_i) = P_o(X_j)$;

and

(ii) If $B \Rightarrow E$ and $P_o(B) = tP_o(E)$, t an irrational number, then there are four infinite sequences of propositions $\{X_i\}$, $\{X_i'\}$, $\{Y_i\}$ and $\{Y_i'\}$, each proposition of which implies E such that:

(iia) $P_o(X_i) \to P_o(B)$ from below and $P_o(Y_i) \to P_o(B)$ from above, as $i \to \infty$,

and for all i,

(iib) $P_o(X_i)$ and $P_o(Y_i)$ are rational multiples of $P_o(E)$,

and

(iic) $P_o(X_i \vee X_i') = P_o(B) = P_o(Y_i\bar{Y}_i')$.

I will now prove that, for all E,

(1.4.8.) If the agent's domain of beliefs is full enough at E, and if $C(E)$, then $(A)(P_o(A/E)) = P_n(A/E))$.

Let E be given, suppose that $C(E)$, and that the agent's domain of beliefs is full enough at E. Since $C(E)$, $P_o(E) \neq 0$ and $P_n(E) \neq 0$. Finally, let A be given. The proof falls exhaustively into two cases:

Case 1: $P_o(AE) = (r/s)P_o(E)$, for integers r, s. Since the agent's domain of beliefs are assumed full enough at E, there is a sequence of propositions $\{X_i\}_{i=1}^{s}$ satisfying (1.4.7.; (ia)–(ic)) with 'AE' substituted for 'B'. By the assumption of $C(E)$ and (1.4.7., (ic)) $P_n(X_i) = P_n(X_j)$ for $1 < i < j < s$. The desired result is now at hand because r of the s propositions of $\{X_i\}_{i=1}^{s}$ give the value for AE as r/s times the value for E both before and after the change of belief. More exactly, $\{X_i\}$ is a logically exclusive and exhaustive partition of E (1.4.7., (ia), (ib)) and for each i, $1 < i < s$, $P_o(X_i) = (1/s)P_o(E)$

and $P_n(X_i) = (1/s)P_n(E)$. So

$$P_o\left(\bigvee_{i=1}^{r} X_i\right) = \frac{r}{s} P_o(E),$$

and

$$P_n\left(\bigvee_{i=1}^{r} X_i\right) = \frac{r}{s} P_n(E).$$

But by assumption of Case 1, $P_o(AE) = (r/s)P_o(E)$. So

$$P_o(AE) = P_o\left(\bigvee_{i=1}^{r} X_i\right),$$

and by $C(E)$,

$$P_n(AE) = P_n\left(\bigvee_{i=1}^{r} X_i\right).$$

Putting the last four lines together we get $P_o(AE) = (r/s)P_o(E)$ and $P_n(AE) = (r/s)P_n(E)$. So

$$P_o(A/E) =_{\text{def}} P_o(AE)/P_o(E) = \frac{r}{s} = P_n(AE)/P_n(E)$$
$$=_{\text{def}} P_n(A/E),$$

which was to be shown.

Case 2: $P_o(AE) = tP_o(E)$, t an irrational number. Again, by the assumption that the agent's beliefs are full enough at E, there are four sequences of propositions, $\{X_i\}$, $\{X_i'\}$, $\{Y_i\}$ and $\{Y_i'\}$ satisfying the conditions of (1.4.7., (iia)–(iic)) with 'AE' substituted for 'B'. By the assumption of $C(E)$ and (1.4.7., (iic)) with 'AE' substituted for 'B' we have, for each i,

$$P_n(X_i \vee X_i') = P_n(AE) = P_n(Y_i\bar{Y}_i').$$

Using the inequalities $P_n(X_i) \leqslant P_n(X_i \vee X_i')$ and $P_n(Y_i\bar{Y}_i') \leqslant P_n(Y_i)$, the last line, and dividing through by $P_n(E)$, we get, for each i,

$$\frac{P_n(X_i)}{P_n(E)} \leqslant \frac{P_n(X_i \vee X_i')}{P_n(E)} = \frac{P_n(AE)}{P_n(E)} = \frac{P_n(Y_i\bar{Y}_i')}{P_n(E)} \leqslant \frac{P_n(Y_i)}{P_n(E)}.$$

By (1.4.7, (iib)), for each i $P_o(X_i)/P_o(E)$ and $P_o(Y_i)/P_o(E)$ are rational numbers; also X_i implies E, and Y_i implies E. So, by case 1, for all i,

$$\frac{P_o(X_i)}{P_o(E)} = \frac{P_n(X_i)}{P_n(E)} \quad \text{and} \quad \frac{P_o(Y_i)}{P_o(E)} = \frac{P_n(Y_i)}{P_n(E)}.$$

By (1.4.7, (iia)) with 'AE' substituted for 'B' and the last line,

$$\frac{P_n(X_i)}{P_n(E)} \rightarrow \frac{P_o(AE)}{P_o(E)} \quad \text{from below as } i \rightarrow \infty \text{ and}$$

$$\frac{P_n(Y_i)}{P_n(E)} \rightarrow \frac{P_o(AE)}{P_o(E)} \quad \text{from above as } i \rightarrow \infty .$$

This together with the one but last line gives $P_o(AE)/P_o(E) = P_n(AE)/P_n(E)$, i.e. $P_o(A/E) = P_n(A/B)$, which was to be shown.

Putting together (1.4.3.), (1.4.4.), and (1.4.8.) it follows for every proposition, E, in the agent's domain of beliefs that if the agent's domain of beliefs is full enough at E, then Cond(E) if and only if $C(E)$. Henceforth I will assume that the agent's domain of beliefs is full enough at every E in his domain of beliefs, and I will call this the limited fullness assumption. Under this assumption it follows from (1.4.3.), (1.4.4.) and (1.4.8.) that

(1.4.9.) For every E in the agent's domain of beliefs Cond(E) if and only if $C(E)$.

The limited fullness assumption seems to me to be a less severe assumption than it will at first appear to the reader. Let E be given and consider any proposition B which implies E and such that $P_o(B) = (r/s)P_o(E)$ for integers r and s. Suppose that there is some randomizing device such as a coin or a die which has outcomes $\{U_i\}_{i=1}^{s}$ which the agent regards as equiprobable and independent of the truth of B before and after he comes to know that E is true. Then the sequence $\{X_i\}_{i=1}^{s}$ which must exist for the agent's beliefs to be full enough at E according to definition (1.4.7.) are just the propositions $\{BU_i\}_{i=1}^{s}$. As for sequences of propositions as described in (1.4.7., (ii)), the limited fullness assumption requires them only in case there is a proposition, B, in the agent's domain of beliefs such that $P_o(B)/P_o(E)$ is irrational; and for a rational agent this will be an exceptional situation. It seems plausible to suppose that this will happen only when there is some operation, such as taking a logarithm or square root or calculating the area of a circle, which is entering into the considerations of the agent. But if some such specific operation is in question, it again seems plausible to suppose that it can be employed to set up certain randomizing devices with outcomes described by the sort of propositions required for limited fullness. It is hard to see how this contention could be supported in general, but I will illustrate it in terms of a specific example. Suppose that there are propositions B and E in the agent's domain of beliefs such that B implies E and $P_o(B)/P_o(E) = 1/\sqrt{2}$. Let k_i be a sequence of rational numbers that converge to $1/\sqrt{2}$ from below; for example, k_i could be taken to be the i'th decimal expansion of $1/\sqrt{2}$ rounded down. For a given i, mark off a segment of length $k_i\sqrt{2}$. This can be done, for example, by constructing a square with side of length k_i and taking the square's diagonal as the segment. Then take a roulette wheel with unit circumference and fashioned with a pointer, which the agent regards as balanced and the outcomes of which the agent regards as independent of the truth of B, both before and after E is found to be true. Color the constructed line segment green and lay it on the circumference of the roulette wheel, and color the remaining portion of the circumference red. Spin the roulette wheel. Then X_i is the proposition that B-and-the-pointer-comes-to-rest-on-the-green-portion-of-the-circumference. X_i' is the proposition that B-and-the-pointer-comes-to-rest-on-the-red-portion-of-the-circumference. To obtain the propositions Y_i and Y_j', let j_i be a sequence that approaches $1/\sqrt{2}$ from above, where j_1 is sufficiently close to $1/\sqrt{2}$. For a given j_i, use geometrical methods to construct line segments of length $(j_i - 1/\sqrt{2})/(1 - 1/\sqrt{2})$. Color this line segment green and lay it along the circumference of a roulette wheel as in the previous case. Spin the wheel. Then Y_i is the proposition that B-or-($\bar{B}$ and E-and-the-pointer-comes-to-rest-on-the-green-portion-of-the-circumference). Y' is the proposition that B-and-E-and-the-pointer-comes-to-rest-on-the-green-portion-of-the-circumference.

Although these existence assumptions are weak, they still involve a considerable degree of idealization. Most of the required randomizing devices do not actually exist, and certainly not infinitely many of them. So real agents do not have most of the propositions describing the outcomes produced by such devices in their actual belief structures. However, the strength of the assumptions can be drastically reduced in the following way. All we really need to assume is the truth of certain counterfactual conditionals. For every B and E such that B implies E and $P_o(B)/P_o(E)$ is irrational, and for every integer n, we need to assume that

> if there were n randomizing devices of the kind described, and $\{X_i\}_{i=1}^{n}$ were in the agent's belief structure, the agent's relative

> degrees of belief $P_o(B)/P_o(E)$ and $P_n(B)/P_n(E)$ would be unaffected either by the existence of the devices or by learning the truth values of the propositions in $\{X_i\}_{i=1}^n$; and $C(E)$ would hold in the expanded belief structure if it held in the original belief structure

and similarly for the propositions in $\{X_i'\}$, $\{Y_i\}$, and $\{Y_i'\}$. Since it is easy to imagine randomizing devices which, if they were to exist, would have the required properties of independence from the agent's beliefs, all such counter-factuals seem quite clearly to be true.

Finally, it is to be remarked that the existence assumptions for the case in which $P_o(B)/P_o(E)$ is rational can be similarly exchanged for the assumption of the truth of corresponding counter-factuals.

1.5. *A Qualitative Principle of Inductive Reasoning and the Justification of Conditionalization*

How might the equivalence of condition C and conditionalization be of help in justifying conditionalization? In the first place the equivalence should help us see to what we are committed when we embrace conditionalization as describing reasonable change of belief, for condition C is, psychologically speaking, a much simpler condition than conditionalization. I say C is simpler because (1) it is a qualitative rather than a quantitative condition on change of belief, (2) it specifies how new beliefs are related to old only in the highly restricted case in which, initially, two beliefs are of equal strength, and (3) this specification is itself highly simple, stating simply that if equal before the change, two beliefs are equal after the change.

In setting out to use the equivalence to justify conditionalization, we should first note that conditionalization on a given proposition E surely does not always describe reasonable change of belief even when the agent comes to know that E is true. Clear cut exceptions are to be found among cases in which E does not cover everything relevant that the agent comes to know in the process of changing beliefs and among cases in which a degree of belief equal to 1 is changed. On the other hand, I doubt that anyone wants to deny that conditionalization describes reasonable change of belief in any circumstances. Surely, in at least some of the highly regimented situations studied by statisticians, conditionalization gives a correct description. So our problem is clearly one of stating the circumstances under which it seems that reasonable change of belief is described by conditionalization. And since condition C is psychologically simpler than conditionalization, it should be easier to single out such circumstances in terms of condition C. Condition C could be used for this purpose in a great variety of ways. What follows is merely one, I think quite conservative such attempt.

I will proceed by first singling out circumstances in which new beliefs are not reasonable unless condition C holds for a given proposition E. By definition, $C(E)$ holds only if the agent moves from a state of doubt about the truth of E to a state of certainty that E is true. Since such certainty seems reasonable only if the agent comes to know E is true, we will look only at circumstances in which the agent comes to have such knowledge. Since it is unclear what new beliefs are reasonable when the agent's initial beliefs are unreasonable, we will restrict the circumstances under consideration to those in which the agent's initial beliefs are reasonable. When we turn to consider reasonable *change* of belief, this restriction will in large measure drop out. Finally reasonable new beliefs often do not seem to arise by conditionalization on E when E does not cover all of the agent's relevant new knowledge; so we should also restrict the circumstances to those in which E satisfies some sort of a total evidence requirement.

These suggestions are captured by the following qualitative principle of inductive reasoning:

P: Let E be any proposition such that

(a) The agent's initial degrees of belief are reasonable.
(b) Initially the agent is unsure of the truth of E.
(c) The agent comes to know that E is true.

and

(d) After coming to know that E is true, any reasons the agent might have which in fact make reasonable or justify changes in other beliefs are either directly given by or included in his new knowledge that E is true; or such reasons indirectly rest on his new knowledge that E is true.

Then for any two propositions A and B, such that

(e) A and B each logically imply E.

and

(f) The agent's initial degree of belief in A and in B are the same;

it is also the case that

(g) The agent's new degrees of belief in A and in B are reasonable only if after coming to know that E is true they continue to be the same.

Several of the terms used in clause (d) need to be explained. I take a reason (of the sort a person may be said to *have*) to be a belief. I will say that a belief whose object is proposition X is directly given by a belief (or, in particular, knowledge) whose object is proposition Y, just in case $X = Y$ or $X = \bar{Y}$. Thus, for example, the agent's knowledge that $\bar{X}$ is false is directly given by his knowledge that X is true. Next, I will say that a belief whose object is proposition X is included in the belief (or, in particular, knowledge) whose object is proposition Y if X is a conjunct of Y or X is the negation of a conjunct of Y. Finally, I will say that a belief whose object is proposition X indirectly rests on a belief (or again, knowledge) whose object is proposition Y just in case the agent has arrived at his belief that X is true by a chain of reasoning whose initial premises are all directly given by or included in his belief (or knowledge) that Y is true.

Note that principle P does not assume that degrees of belief can be quantized. It assumes only that beliefs can be qualitatively ordered as to strength; and for any two beliefs, one will be found to be stronger than the other or they will be found to be the same in strength.

It is easy to check that, if principle P is true, and if conditions (a)–(d) hold true of a proposition E, then the new beliefs are all reasonable only if $C(E)$. Since $C(E)$ if and only if change of belief is described by conditionalization on E, and since by definition (0.1.1.), if the initial beliefs are reasonable, *change* of belief is reasonable only if the new beliefs are reasonable, it follows that if principle P is true, and if conditions (a)–(d) hold true of a proposition E, then change of belief is reasonable only if it is described by conditionalization on E.

Principle *P* also applies, as follows, to certain cases in which (a) fails. Assume

(1.5.1.) *P* is true, (b)–(d) hold true of *E*, and (a) is false.

Then the foregoing argument sustains the following counter-factual conditional:

(1.5.2.) If the initial beliefs (the ones the agent in fact held) had been reasonable, the new beliefs would have been reasonable only if they arose from the old beliefs by conditionalization on *E*.

Assume further that

(1.5.3.) Had the initial beliefs (the ones the agent in fact held) been reasonable, then *some* new overall set of beliefs would have been reasonable.

Then it follows, from (1.5.2.) and (1.5.3.) that

(1.5.4.) If the initial beliefs (the ones the agent in fact held) had been reasonable, then new beliefs arising from the old by conditionalization on *E*, and only these, would have been reasonable.

Finally, consider the assumption that

(1.5.5.) Both before and after the change of belief, the agent has a high, reasonable degree of belief that his initial beliefs were reasonable.

It follows from (1.5.4.), (1.5.5.) and the definition of reasonable *change* of belief (0.1.1.) that a *change* of belief arising from the old beliefs by conditionalization on *E* is a reasonable change.

In summary, if principle *P* is true, and if conditions (b)–(d) hold true of a proposition *E*, then

(i) if (a) holds, change of belief by conditionalization on *E* is a reasonable change (and the only reasonable change), if any change is reasonable.

and

(ii) if (a) fails, but assumptions (1.5.3.) and (1.5.5.) hold, then by definition (0.1.1.), change of belief by conditionalization on *E* is reasonable.

Thus barring situations in which no change is or would be reasonable, and situations in which the agent fails to have considerable reasonable confidence in the rationality of his original beliefs, if principle *P* is true, knowledge that (b)–(d) hold true of a proposition *E* provides substantial justification for change of belief by conditionalization on *E*.

But is principle *P* true? I think the principle is intuitively plausible, it is interesting in its own right, and it merits both critical scrutiny and efforts to derive it from still more obvious principles of non-deductive reasoning. But for the moment I can defend it only by offering some examples, which the reader is invited to multiply, and by examining some general arguments which some will think to advance against it.

Suppose two men are going to race and the agent has equal strength of belief in the propositions *A*, that the first man wins, and *B*, that the second man wins. The agent is unsure of the truth of *E*, the proposition that one of the men wins, because he recognizes that the race might be called off or might result in a tie. The agent now learns (and learns no more than) that *E* is true; the race was successfully completed and did not result in a tie. Under these conditions it would be absurd for him now to shift his beliefs so that he is rather more confident in *A* than in *B* or *B* than *A*.

To present another example, suppose that five tosses of a given coin are planned and that the agent has equal strength of belief for two outcomes, both beginning with *H*, say the outcomes *HTTHT* and *HHTTH*. Suppose the first toss is made, and results in a head. If all that the agent learns is that a head occurred on the first toss it seems unreasonable for him to move to a greater confidence in the occurrence of one sequence rather than another. The only thing he has found out is something which is logically implied by both propositions, and hence, it seems plausible to say, fails to differentiate between them.

This second example might be challenged along the following lines: The case might be one in which initially the agent is moderately confident that the coin is either biased toward heads or toward tails. But he has as strong a belief that the bias is the one way as the other. So initially he has the same degree of confidence that *H* will occur as that *T* will occur on any given toss, and so, by symmetry considerations, an equal degree of confidence in *HTTHT* and *HHTTH*. Now if *H* is observed on the first toss it is reasonable for the agent to have slightly more confidence that the coin is biased toward heads than toward tails. And if so it might seem he now should have more confidence that the sequence should conclude with the results *HTTH* than *TTHT* because the first of these sequence has more heads in it than tails. However, there is more to be said. Consider that nothing has happened to cast doubt on the agent's (assumed reasonable) original opinion that the coin is biased one way or the other, and that *HTTH* is a relatively unlikely occurrence under either the hypothesis of bias towards heads or the hypothesis of bias towards tails. It is true that, after the observation, *TTHT* seems less likely than, say, *HHTH*; but the evidence will not have fully convinced the agent that bias towards tails is to be ruled out. Indeed, in this example, after the observation the agent will have only slightly more confidence that the bias is toward heads rather than tails. Insofar as the agent continues to believe that there is a bias one way or the other, both *TTHT* and *HHTH* should seem more likely than a sequence like *HTTH*. Whether or not these qualitative considerations exactly balance out cannot be decided without, at least implicitly, opting for some quantitative principle for change of belief. And since an argument for or against such a principle is precisely what is at issue in examining conditionalization, we must, on pain of begging the question, leave these competing qualitative considerations to compete inconclusively.

I turn now from specific examples to a general argument which might be put forward against principle P. On the hypythetico-deductive account of confirmation, a theory T is said to be confirmed by the observations it entails. But it is now widely agreed that of two theories, T_1 and T_2, both of which imply all the observations which have in fact been made, one may be better confirmed than the other. One might be tempted to conclude that a counter example to our principle can be found in some conjunction O, describing all performed observations entailed by T_1 and T_2 where O confirms one of the theories more than the other. But to be a counter-example such a case would have to be one in which both T_1 and T_2 commanded equal reasonable confidence before the observations and in which the final difference in confirmation cannot be accounted for by factors besides the entailed observations, such as simplicity or explanatory power. Careful examination of specific examples suggests that, when the observational evidence is the same, differences in final confirmation can always be attributed to differences in initial confirmation or other considerations extraneous to the observations. These claims are born out particularly clearly in the simple case of Goodman's green hypothesis, that all emeralds are green, and grue hypothesis, that all emeralds are grue. Both hypotheses entail the observational instantial evidence of heretofore observed green emeralds. Yet the green hypothesis is to be counted as the better confirmed of the two. But for this case to provide a counter example to principle *P* one would have to establish the claims that if no emeralds had been observed the two hypotheses would command equal degrees of reasonable belief and that the observations confirmed the green hypothesis but not the grue hypothesis or that the observations confirmed the first more than the second. The first claim is patently false. As for the second, Goodman and others often say that only the green hypothesis is confirmed by emeralds observed to be green, but it has never been argued that the difference in final confirmation of the two hypotheses is to be accounted for by the differential bearing of the observations themselves rather than other considerations. Indeed, when viewed as part of the overall problems of confirmation it becomes at least as plausible to say that the difference in the final status of the hypotheses is wholely attributable to the difference in the hypotheses before observations are taken into account. (This is argued in [11], pp. 234–7 and *passim.*)

1.6. *On Observation*

If principle *P* is accepted, we have an interesting specification of a wide range of circumstances in which change of belief by conditionalization can be justified. However, this specification is really valuable only if we can ascertain when the conditions (b)–(d) of principle *P* are met; and this might be found to be difficult. Often, when change of belief by conditionalization seems appropriate, the change appears to originate in the agent's making a perceptual observation. Thus it seems plausible to suppose that some of the conditions (b)–(d) of principle *P* might be illuminatingly tied to an analysis of observational knowledge. In this Section I propose a partial analysis of observation which will allow us to connect observation with conditions (c) and (d) of principle *P*. The following section will give the details of the connection. The material presented here will focus on the connection between observation and conditionalization, and will not deal with independently existing problems in the analysis of observational knowledge.

I will use the term 'strict observation' to refer to any event satisfying the following conditions:

(1.6.1.)(i) There is a non-empty finite set of propositions $\{A_i\}_{i \in I}$ such that at the time of or during the occurrence of the event the agent's degree of belief in each of these propositions changes.

(ii) For each $i \in I$, the agent's change of belief in A_i and in $\bar{A}_i$ is caused by the environment's effects on the agent's sense organs.

(iii) For each $i \in I$, the agent's change of belief in A_i and in $\bar{A}_i$ takes place without conscious inference or reasoning of any kind.

(iv) For any proposition B which is not in the set $\{A_i\}_{i \in I}$ or in the set $\{\bar{A}_i\}_{i \in I}$ and for which the agent's degree of belief changes, the change of belief in B is not both caused by the environment's effects on the agent's sense organs and also not the result of conscious reasoning of any kind (i.e., the conditions of (ii) and (iii) do not both hold).

(v) Conditions are such that for each A_i, $i \in I$, the agent's new belief in A_i is reasonable, and in particular counts as knowledge that A_i is true.

For any strict observation the proposition $A = \bigwedge_{i \in I} A_i$ will be called the observation's observed proposition.

The importance of this definition lies in the fact that many, perhaps most, of the events we commonly refer to as observations seem to satisfy the conditions for being a strict observation, with one qualification, to be discussed below. When I draw the blinds in the morning and, blinking in the sunlight, observe that the sun is shining, the sunlight striking my eyes under those conditions causes me to believe that the sun is shining. Though, conceivably, I might later revise my opinion, I have no choice when I first look; upon looking I am caused, willi nillie, to believe that the sun is shining. Nor, in the usual case, does any form of conscious inference accompany this change of state of belief. I look and I am caused to believe that the sun is shining and that, from the point of view of my conscious rational processes, is all there is to it. Similarly, in ordinary circumstances, when I observe a flipped coin to come up heads, the visual pattern which is presented to my eyes causes me to believe, without conscious inference and without choice, that the coin has come up heads. Many more recondite cases are also to be described this way: the practiced archeologist looking at a chipped piece of stone, may be caused by the visual pattern to come to believe that the stone is an artifact. Ordinarily he will do so without conscious inference, even though the unpracticed archeology student, in the same circumstances, may come to the same belief only deliberately and as a result of considerable conscious reasoning. What a man is able to observe strictly will often depend on skills acquired by practice and training.

The reservation in the claim that many cases of ordinary observation constitute strict observations lies in the condition that the new beliefs in the propositions, A_i, $i \in I$ constitute knowledge. I have been assuming throughout Part I, that knowledge involves certainty and that certainty is analyzable within a bayesian framework only as degree of belief equal to 1. But it seems unrealistic to suppose that people often or even ever have degree of belief of 1 in a proposition, or that any such belief is reasonable. Furthermore, even if some observations result in knowledge, clearly many do not. That most ordinary observations constitute strict observations as defined here seems clearly to be an idealization which is both complementary to and on the same sort of footing as the idealization that reasonable change of belief often takes place by conditionalization on some proposition, E, which the agent has come to know to be true. We will continue in both these idealizations here and dispose of them in Part II.

1.7. *Strict Observation and the Conditions for Conditionalization*

We have seen in Section 1.5. that conditionalization on a proposition E can be justified by appeal to principle P when it is known that conditions (b)–(d) of principle P are satisfied. And in Section 1.6. we raised the question of now it can be known that these conditions hold. Condition (b) is not particularly problematic. And we can now make use of our definition of strict observation in specifying circumstances in which (c) and (d) hold.

Before examining conditions (c) and (d), we need several preliminary definitions and remarks. Let us say that an agent's beliefs are *stable* if none of his beliefs constitute reasons which would make reasonable or justify changes in the degree to which he believes other propositions. Clearly (d) holds only if the agent's beliefs are initially stable. Moreover, reasonable change of belief will not in general be given by conditionalization when the initial beliefs are not stable. Consequently we need to require stability of initial beliefs for condition (d) to hold. Let us call a degree of belief, $P_n(R)$, after the agent comes to know that E is true a *new (degree of) belief* if $P_n(R) \neq P_o(R)$, where $P_o(R)$ is the degree of belief the agent had before coming to know E is true. If $P_n(R)$ also constitutes a reason the agent has, which in fact makes reasonable or justifies his changing other degrees of belief, we will also call $P_n(R)$ a *new reason*. Next, we shall say that

> Proposition X is a *particle* of proposition Y if and only if $X = Y$ or $X = \bar{Y}$ or X is a conjunct of Y or X is the negation of a conjunct of Y.

I said in Section 1.5. that if $X = Y$ or if $X = \bar{Y}$, the agent's belief in X was directly given by his belief in Y; and if X is a conjunct of Y, or the negation of a conjunct of Y, the agent's belief in X was included in his belief in Y. Consequently, for any particle X, of Y, $P_n(X)$ is included in or given by $P_n(Y)$. In particular for any particle, X, of Y, the agent's knowledge that X is true (or false) is directly given by or included in his knowledge that Y is true. Finally, I shall assume, which I hope is obvious, that an uncaused, unreasoned new belief (if such a thing exists) is not reasonable. Furthermore an unreasonable new belief cannot in fact make reasonable or justify changes in other beliefs. So an unreasonable new belief cannot be a new reason.

I will now argue that, if the agent's initial beliefs are stable and if he makes a strict observation with observed proposition E, conditions (c) and (d) of principle P hold for E. Assume initial stability and the occurrence of a strict observation with observed proposition $E = \bigwedge_{i \in I} A_i$. Assuming that if an agent (assumed throughout to be an ideal logician) knows the conjuncts of a proposition, he knows the proposition, it follows from condition (v) of the definition of strict observation (1.6.1.) that condition (c) holds true of E. We next show that condition (d) holds for E. Since the initial beliefs are assumed to be stable, any reason the agent might have which in fact makes reasonable or justifies changes in other beliefs must be a new reason. Let $P_n(R)$ be any such new reason. If R is a particle of E, $P_n(R)$ is directly given by or included in the agent's knowledge of E. So we have only to consider R's which are not particles of E. Assume R is not a particle of E. $P_n(R)$ is either consciously reasoned or not consciously reasoned. Let us consider unreasoned $P_n(R)$ first. If $P_n(R)$ is also uncaused, it is not reasonable, and so, as argued above it is not, after all, a new reason. If $P_n(R)$ is caused as well as unreasoned, it is, by the definition of strict observation, after all, a particle of E. So we have left to consider only R's such that $P_n(R)$ is reasoned. Since the agent's beliefs are assumed to be initially stable $R_n(R)$ must be reasoned on the basis of new reasons $\{P_n(R_j)\}_{j \in J}$. If a $P_n(R_j)$ is unreasoned, as before it must be caused and so a particle of E. If an $P_n(R_j)$ is reasoned the argument applies again as it did in the case of the reasoned $P_n(R)$. Assuming that chains of reasoning are finite, such reapplications of the argument must come to an end in a case in which the basis of reasoning contains only particles of E. Hence *all* the original premises which form the basis of reasoning for $P_n(R)$, are particles of E, used at one or another stage of the reasoning in support of $P_n(R)$; and $P_n(R)$ indirectly rests on the agent's knowledge that E is true, as 'indirectly rests on' was explained in Section 1.5.

1.8. *Summary and Concluding Remarks to Part I*

In this part I have briefly remarked on frequentist and Dutch Book Arguments in support of change of belief by conditionalization. I have shown that change of belief by conditionalization is equivalent to a psychologically simpler qualitative condition on change of belief. I have used this condition in developing, P, a qualitative principle of inductive reasoning, which can be used to justify change of belief by conditionalization when the conditions of application of the principle are met. And finally, I have presented a partial analysis of observation and used this analysis in further specifying the conditions of application of principle P. Residual problems include that of finding other conditions, if there are any, under which the conditions of application of principle P apply to a proposition E, and the problem of further clarifying the conditions under which an event constitutes a strict observation. Finally, we still have the problem of generalizing our results to enable us to describe change of belief which does not require the agent to have either degree of belief of 1 in any proposition or the sort of knowledge and certainty which seem to be describable only in terms of such absolute degrees of belief. This last problem will be the subject of Part II.

PART II

2.1. *Generalized Conditionalization*

To present the present topic, it will help to have at hand several examples of observations which are not strict and changes of belief which cannot be described using the methods of Part I.

Case 1: The agent has a piece of cloth which he knows is either brown (B) or green (G), and is either dyed with a natural (N) or a synthetic (S) dye. Initially the agent's degree of belief are

$$\begin{array}{lll} P_o(B)=\frac{1}{3}, & P_o(S/B)=\frac{1}{3}, & P_o(SB)=\frac{1}{9}, \\ P_o(G)=\frac{2}{3}, & P_o(N/B)=\frac{2}{3}, & P_o(NB)=\frac{2}{9}, \\ P_o(S)=\frac{5}{9}, & P_o(S/G)=\frac{2}{3}, & P_o(SG)=\frac{4}{9}, \\ P_o(N)=\frac{4}{9}, & P_o(N/G)=\frac{1}{3}, & P_o(NG)=\frac{2}{9}. \end{array}$$

The Venn diagram is

	B	G
S	$P_o(SB)=\frac{1}{9}$	$P_o(SG)=\frac{4}{9}$
N	$P_o(NB)=\frac{2}{9}$	$P_o(NG)=\frac{2}{9}$

The agent observes the cloth by candle light, so that he cannot see the color very clearly. As a result, he comes to have new degrees of belief about the color:

$$P_n(B)=\tfrac{2}{3}, \qquad P_n(G)=\tfrac{1}{3},$$

but the conditional degrees of belief remain unaltered:

$$P_o(S/B)=P_n(S/B),\ P_o(N/B)=P_n(N/B),\ P_o(S/G)=P_n(S/G),$$

$$\text{and}\quad P_o(N/G)=P_n(N/G)$$

We assume that the change of belief is reasonable.

Case 2: The agent had the same initial degrees of belief as in Case 1, but this time he observes the cloth in broad daylight and sees quite clearly that it is brown. However, the circumstances are such that his observation causes him to shift his belief that the dye is synthetic to $\frac{2}{3}$. His new degrees of belief are

$$P_n(B) = 1 \qquad P_n(S) = \tfrac{2}{3},$$
$$P_n(G) = 0 \qquad P_n(N) = \tfrac{1}{3}.$$

We assume that neither we nor the agent can describe any special quality of the brown hue which causes this shift, so that there is no further proposition available in terms of which the problem could be redescribed allowing the change of belief about the dye to be characterized as arising by conditionalization. We assume the agent's change of belief to be rational, as might be the case if he has wide ranging but untutored experience in judging dyes.

These cases exemplify observation which are not strict and changes of belief which cannot be described by conditionalization. But the changes of belief can be characterized by a generalized form of conditionalization suggested by Richard Jeffrey ([6] pp. 157–63). Let capital letters range over propositions in the domain of the agent's belief function, and suppose that the agent changes his belief from the original function, P_o, to the new function P_n. Suppose $\{E_i\}_{i \in I}$ is a set of propositions such that

(2.1.1.)(a) $\bigvee_{i \in I} E_i$ is logically true and $E_i E_j$ is logically false $i \neq j$, $i \in I$,

(b) If $P_o(E_i) = 0$, then $P_n(E_i) = 0$, $i \in I$,

(c) If $P_o(E_i) \neq 0$ and $P_n(E_i) \neq 0$, then for all A, $P_o(A/E_i) = P_n(A/E_i)$, $i \in I$.

(Henceforth the index variable i will always be assumed to be in index set I, and explicit reference to I will be omitted when no confusion will result.) If these conditions are met it follows immediately that, for all A,

$$(2.1.2.) \quad P_n(A) = \sum_{\substack{i \\ P_o(E_i) \neq 0}} P_o(A/E_i)\, P_n(E_i).$$

If a set of propositions $\{E_i\}$ meets conditions (2.1.1.) for a change of belief from P_o to P_n we will say that the change *originates* in the set of propositions $\{E_i\}$, and we will refer to any change of belief which can be described by the corresponding (2.1.2.) as a change of belief described by, determined by, or arising by *generalized conditionalization*. To illustrate, in Case 1, the conditions are satisfied for saying that the change of belief originates in the propositions B and G. The new degree of belief in S is

$$\begin{aligned} P_n(S) &= P_o(S/B)P_n(B) + P_o(S/G)P_n(G), \\ &= \tfrac{1}{3}\cdot\tfrac{2}{3} + \tfrac{2}{3}\cdot\tfrac{1}{3}, \\ &= \tfrac{4}{9}. \end{aligned}$$

Henceforth the form of conditionalization discussed in Part I and defined by the condition Cond (E) will be called *strict conditionalization.* Strict conditionalization is obviously a special case of generalized conditionalization.

We have strong reasons for seeking to describe reasonable change of belief by generalized rather than strict conditionalization. Strict conditionalization requires description of the agent as coming to have a perfect degree of belief in a proposition, a degree of belief which cannot be altered by conditionalization within the bayesian framework, and which commits one to betting on the proposition at arbitrarily risky odds. Rarely, if ever, is it reasonable to have such a perfect degree of belief. To be sure, reasonable degrees of belief are often close to one, in which case strict conditionalization can be used as an idealization or useful approximate description. But in such a situation generalized conditionalization offers a more accurate and less idealized description of what change of belief is reasonable. Moreover, in situations like Case 1, strict conditionalization does not offer even an approximate description of the change of belief. This is so in such cases unless some intervening proposition, perhaps describing something like 'sense data', can be found. In Case 1, for example, it might be suggested that after all, the agent comes to have a perfect or near perfect degree of belief in a proposition describing the way the data appeared to him in the dim light, and his degree of belief in propositions B, G, S, and N can be described by conditionalization on this proposition. However, such attempts to make strict conditionalization applicable to this kind of situation seems to me entirely gratuitous. In most such cases our language and that of the agent does not include the sentences with which the required propositions can be specified. And there does not seem to be any other, perhaps indirect way of singling out the required propositions other than question begging descriptions of the sort, "The proposition in terms of which the change of belief can be described by strict conditionalization". Furthermore, in view of the strength of present arguments against a sense data account of human perception, there is no reason to expect that future advances in our understanding of perceptual processes will present us with well confirmed theories according to which agents do after all entertain belief in propositions of the sort required to make strict conditionalization applicable to cases like Case 1.

While the foregoing remarks show generalized conditionalization to be an attractive refinement of strict conditionalization, it remains to be shown how change of belief by generalized conditionalization can be justified. And there might seem to be a problem with appeal to generalized conditionalization as a means of describing reasonable change of belief, for almost *any* change of belief can be characterized as one which takes place by generalized conditionalization orginating in some set of propositions. In Case 2 above, the change can be characterized as one originating in the set of propositions *SB*, *NB*, *SG*, and *NG*, as can any other change from this original set of beliefs. With one class of exceptions, if the propositions of a problem situation include a finite basic set, that is a set of logically exhaustive, mutually exclusive propositions in terms of which all the other propositions can be expressed as truth functional combinations, then any change of belief can be characterized by generalized conditionalization orginating in the basic set. The exceptions are cases in which an initial belief of degree zero or one changes. If the propositions of a problem situation cannot be characterized in terms of a basic set, because the problem situation includes an infinite number of logically independent propositions, the propositions of the problem situation can in all interesting cases still be characterized to an arbitrarily set degree of accuracy by a basic set; and with the exceptions mentioned above, any change of belief can be characterized to within an arbitrarily set degree of accuracy by generalized conditionalization originating in such a basic set.

Since virtually all changes of belief can be characterized, either exactly or to within an arbitrarily set degree of accuracy, as changes by generalized conditionalization, it might seem impossible to use generalized conditionalization to distinguish reasonable from unreasonable change of belief. This conclusion is unwarranted if situations of the following sort may arise. Suppose that a change of belief from P_o to P_n originates in the set $\{E_i\}$ nontrivially, in the sense that at least one proposition, A, in the domain of the agent's belief function cannot be expressed as a truth functional combination of the members of $\{E_i\}$ nor in any relevant sense can A be approximated by such a truth function. Suppose that it can be independently established that the agent's change of belief for the members of $\{E_i\}$ is reasonable. For example, this might be done by appeal to facts about an observation. Finally, suppose it can be argued that, under the circumstances, if change of belief in the members of $\{E_i\}$ is reasonable, no overall change of belief is reasonable except the one described by generalized conditionalization originating in the set $\{E_i\}$. Under such conditions one may conclude that only this change is reasonable.

The following sections will show how change of belief by generalized conditionalization may be justified by developing an argument following the pattern of the kind sketched above. This is accomplished by a straightforward generalization of the contents of Sections 1.4.–1.7. I will omit arguments which are largely repetitious of ones given in Part I.

2.2. *Equivalence of Generalized Conditionalization and a Qualitative Condition on Change of Belief*

Suppose again that a change in belief takes place, that P_o describes the agent's beliefs before the change, and that P_n describes his beliefs after the change. Let '$\Rightarrow$' mean 'logically implies'. A set $\{E_i\}$ will always be understood to be a set of at least two propositions satisfying condition (2.1.1. (a)), that is, a set of logically exhaustive and mutually exclusive propositions. Such a set will henceforth be called a *partition*. We now define condition *GC* applying to sets $\{E_i\}$, by the following generalization of condition *C* of Section 1.4.:

$$\text{(2.2.1.)}\quad GC(\{E_i\}) \equiv_{\text{def}} (\forall i)[(P_o(E_i) = 0 \rightarrow P_n(E_i) = 0)\ \& \ (\forall A)(\forall B)((A \Rightarrow E_i \,\&\, B \Rightarrow E_i \,\&\, P_o(A) = P_o(B) \rightarrow \rightarrow P_n(A) = P_n(B))].$$

As in the case of condition *C*, condition *GC* has been stated in terms of quantitative degrees of belief for convenience of the following proofs. But the conditions is really of a qualitative nature, since each clause is stated only in terms of equality of beliefs or the condition of a belief having an extremum value.

Next we define *G*-Cond $(\{E_i\})$ as the appropriate generalization of Cond (E) of Section 1.4.:

$$\text{(2.2.2.)}\quad G\text{-Cond}(\{E_i\}) \equiv_{\text{def}} (\forall i)\,(P_o(E_i) = 0 \rightarrow P_n(E_i) = 0)\ \& \ (\forall A)\,\Big(P_n(A) = \sum_{\substack{i \\ P_o(E_i) \neq 0}} P_o(A/E_i)\, P_n(E_i)\Big)$$

I will now show that, under the limited fullness assumption described in Part I,

(2.2.3.) For any partition, $\{E_i\}$, $GC(\{E_i\})$ if and only if G-Cond$(\{E_i\})$.

Let a partition, $\{E_i\}$, be given. First I show that if G-Cond $(\{E_i\})$, then $GC(\{E_i\})$. Suppose G-Cond$(\{E_i\})$. Let A and B be any two propositions and E_j a member of $\{E_i\}$ such that $A \Rightarrow E_j$ and $B \Rightarrow E_j$, and $P_o(A) = P_o(B)$. By the assumption of G-Cond$(\{E_i\})$,

$$\text{(2.2.4.)}\quad P_n(A) = \sum_{\substack{i \\ P_o(E_i) \neq 0}} P_o(A/E_i)\, P_n(E_i),$$

$$P_n(B) = \sum_{\substack{i \\ P_o(E_i) \neq 0}} P_o(B/E_i)\, P_n(E_i).$$

Since, by assumption $A \Rightarrow E_j$ and $B \Rightarrow E_j$, and since the members of $\{E_i\}$ are logically incompatiable, $P_o(A/E_i) = 0$ if it exists and $i \neq j$ and $P_o(B/E_i) = 0$ if it exists and $i \neq j$. Consequently, if $P_o(E_j) = 0$, (2.2.4.) gives $P_n(A) = P_n(B) = 0$. If $P_o(E_j) \neq 0$, then (2.2.4.) gives

$$\begin{aligned} P_n(A) &= P_o(A/E_j)P_n(E_j), \\ &= P_o(A)P_n(E_j)/P_o(E_j), \quad \text{since} \quad A \Rightarrow E_j, \end{aligned}$$

and

$$\begin{aligned} P_n(B) &= P_o(B/E_j)P_n(E_j), \\ &= P_o(B)P_n(E_j)/P_o(E_j), \quad \text{since} \quad B \Rightarrow E_j. \end{aligned}$$

But, by assumption $P_o(A) = P_o(B)$. So $P_n(A) = P_n(B)$. This completes the proof that if G-Cond$(\{E_i\})$, then $CG(\{E_i\})$.

Next we turn to the proof of the converse. Assume $GC(\{E_i\})$. It follows that

(2.2.5.) $(\forall i)[P_o(E_i) = 0 \rightarrow P_n(E_i) = 0]$.

First we need to show, on the assumption of (2.2.5.) that if

(2.2.6.) $(\forall i)[(P_o(E_i) \neq 0 \,\&\, P_n(E_i) \neq 0) \rightarrow (A)(P_o(A/E_i) = P_n(A/E_i))]$

then

$$\text{(2.2.7.)}\quad (\forall A)\Big(P_n(A) = \sum_{\substack{i \\ P_o(E_i) \neq 0}} P_o(A/E_i)\, P_n(E_i)\Big).$$

Assume (2.2.6.). Let A be an arbitrary proposition.

$$(2.2.8.)\quad P_n(A) = \sum_i P_n(AE_i),$$
$$= \sum_{\substack{i \\ P_n(E_i) \neq 0}} P_n(AE_i),$$
$$= \sum_{\substack{i \\ P_n(E_i) \neq 0}} P_n(AE_i) P_n(E_i)/P_n(E_i),$$
$$= \sum_{\substack{i \\ P_n(E_i) \neq 0}} P_n(A/E_i)\, P_n(E_i).$$

By (2.2.5.) if $P_n(E_i) \neq 0$, then $P_o(E_i) \neq 0$. And by the assumption of (2.2.6.), if $P_n(E_i) \neq 0$ and $P_o(E_i) \neq 0$, then $P_o(A/E_i) = P_n(A/E_i)$. Consequently (2.2.8.) gives

$$(2.2.9.)\quad P_n(A) = \sum_{\substack{i \\ P_n(E_i) \neq 0}} P_o(A/E_i)\, P_n(E_i).$$

The constraint $P_n(E_i) \neq 0$ has no effect, except, together with (2.2.5.) to guarantee that the terms of the sum exist. So (2.2.9.) gives

$$P_n(A) = \sum_{\substack{i \\ P_o(E_i) \neq 0}} P_o(A/E_i)\, P_n(E_i).$$

This concludes the proof of (2.2.7.) from (2.2.6.) and (2.2.5.).

Since (2.2.7.) follows from (2.2.6.) and (2.2.5.), and (2.2.5.) follows from $GC\{E_i\}$, we need only prove (2.2.6.) from $GC\{E_i\}$ to complete the proof of G-Cond$(\{E_i\})$ from $CG(\{E_i\})$. This final step requires the assumption that, for each i, the agent's domain of beliefs is full enough at E_i (definition 1.4.7.); and as in Part I, I will assume that the agent's domain of beliefs is full enough at each proposition in the domain. Under this assumption, (2.2.6.) is seen to follow from $GC(\{E_i\})$ as an immediate consequence of (1.4.8.). It need only be remarked that although $C(E)$, which appears in the antecedent of (1.4.8.), includes the conjuncts $0 < P_o(E) < 1$ and $P_n(E) = 1$, only the weaker antecedents $P_o(E) \neq 0$ and $P_n(E) \neq 0$, corresponding to the antecedents of (2.2.6.), were used in the proof of (1.4.8.).

2.3. *Generalization of Principle P, and the Justification of Generalized Conditionalization*

The equivalence of generalized conditionalization and condition GC should help us to understand what we are committed to when we accept generalized conditionalization, originating in a given set of propositions, as describing reasonable change of belief. The reasons for this claim are quite the same as the reasons for the parallel claim in the case of strict conditionalization and condition *C*. Again, the generalized equivalence might be exploited in many ways. I present here a straightforward generalization of principle *P* and the use to which the new principle may be put. Corresponding to the case of strict conditionalization, generalized conditionalization on $\{E_i\}$ clearly does not describe a reasonable change of belief when $\{E_i\}$ does not cover everything relevant the agent comes to know in changing beliefs or when the agent changes a belief of degree 0 or 1. Hence, requirements of total evidence and of preservation of extremum degrees of belief in the propositions of $\{E_i\}$ are needed in the conditions of application of the principle.

The important step in generalizing principle *P* is to replace the characterization of the agent as coming to have new knowledge of a proposition *E* with the characterization of the agent as coming to have new reasonable degrees of belief in the propositions of a set $\{E_i\}$ of logically exhaustive and mutually exclusive propositions. Incorporating this change results in principle *PG*:

PG: Consider any partition (a set of at least two propositions satisfying (2.1.1.a)) $\{E_i\}$ and any change of belief such that

(a) The agent's initial degrees of belief are all reasonable.

(b) For any E_i, if the agent is certain that E_i is false before the change, then he is certain that E_i is false after the change.

(c) The agent's strength of belief in at least one of the propositions E_i changes, and after the change the agents beliefs in E_i is reasonable for each *i*.

and

(d) After the change of belief, any reasons the agent might have which in fact make reasonable or justify changes in belief in any proposition $A \notin \{E_i\}$ are beliefs whose objects are propositions in $\{E_i\}$ or disjunctions of these propositions; or else such reasons indirectly rest on his beliefs in the E_i, or their disjunctions.

Then, for any two propositions A, and B such that

(e) There is an i such that A and B each logically imply E_i.

and

(f) The agent's initial strength of belief in A and in B are the same.

It is also the case that

(g) The agent's new beliefs in A and in B are reasonable only if after the change of belief his strength of belief in A and in B continue to be the same.

As in the case of principle P, I take a reason (of the sort a person may be said to *have*) to be a belief; and I say that a belief whose object is proposition X indirectly rests on beliefs whose objects are the propositions in $\{Y_j\}_{j \in J}$ just in case the agent has arrived at his belief in X by a chain of reasoning whose initial premises are among his beliefs in the propositions in $\{Y_j\}_{j \in J}$. In examining condition (d), note that our general assumption that the agent's beliefs are, at any given time, described by a probability function implies that the agent's degrees of belief in the propositions in $\{E_i\}$ fully determine the value of his degrees of belief in the disjunctions of these propositions. As in the case of principle P, principle PG is a purely qualitative principle of inductive reasoning.

I believe that the acceptability of principle PG is much on the same footing as that of principle P; considerations advanced for or against P carry over to the case of PG. I will illustrate briefly by generalizing on the first example given in support of principle P. The agent is a spectator at a race between two men. Initially he has equal strength of belief in the proposition A, that the first man wins and B, that the second man wins. E is the proposition that one of the men wins, and $\{E_i\}$ is the set $\{E, \bar{E}\}$. The agent initially has a much stronger belief in E than in $\bar{E}$. The sprinters have been neck and neck from the start, and as they burst through the tape the hot dog vender passes through the agent's line of vision so he cannot see the finish. The announcer informs the spectators that he could not see who, if either, finished first and so he is waiting for the judges verdict. Under these circumstances the agent's strength of belief in $\bar{E}$ increases; but it would be absurd for him to shift his beliefs so that he is now more confident in A than in B or B than in A.

Continuing the parallel to the development of Section 1.5., it is easy to check that, if principle *PG* is true, and if conditions (a)–(d) hold true for a partition, and if certainty in the falsehood of a proposition A is interpreted by setting $P(A) = 0$, then the new beliefs are all reasonable only if $GC(\{E_i\})$. This fact, together with the equivalence of conditions *GC* and *G*-Cond can be applied in an exact repetition of the argument given in Section 1.5. which links these facts to the definition of reasonable *change* of belief (0.1.1.). Since this argument changes in none of its details it need not be repeated here. The conclusion is as follows:

> If principle *PG* is true and if conditions (b)–(d) of principle *PG* hold true for a change of belief and a partition $\{E_i\}$, then

(i) if (a) of principle *PG* holds, change of belief by generalized conditionalization originating in the set $\{E_i\}$ is a reasonable change, and the only reasonable change, if any change is reasonable.

and

(ii) if (a) fails, but assumptions (1.5.3.) and (1.5.5.) hold then change of belief by generalized conditionalization originating in $\{E_i\}$ is reasonable.

Thus, again in parallel to the case of strict conditionalization, barring situations in which no change is or would be reasonable, and situations in which the agent fails to have considerable reasonable confidence in the rationality of his original beliefs, if principle *PG* is true, reasonable belief that (b)–(d) hold true for a set $\{E_i\}$ provides substantial justification for change of belief by generalized conditionalization originating in $\{E_i\}$.

2.4. *Generalized Observation*

As in the case of principle *P*, principle *PG* provides an interesting specification of a wide range of circumstances in which change of belief by generalized conditionalization can be justified. But the specification is really useful only if we can ascertain when the conditions (b)–(d) of principle *PG* are met. And again, to aid in this task, I propose here a generalization of the analysis of observation presented in Section 1.5. which avoids commitment to observational knowledge. As before, the analysis is partial and leaves aside independently existing problems in the analysis of observation.

I will use the term 'G-observation' to refer to any event satisfying the following conditions:

(i) There is a partition $\{E_i\}$ (a set of at least two propositions which satisfy condition (2.1.1., *a*) i.e. they are logically exhaustive and mutually exclusive), such that at the time of or during the course of the event the agent's strength of belief in at least one of the E_i changes.

(ii) For each i, if the agent's degree of belief in E_i changes, the agent's new degree of belief in E_i is caused by the environment's effects on the agent's sense organs.

(iii) For each i, if the agent's degree of belief in E_i changes, the agent's change of belief in E_i takes place without conscious reasoning of any kind.

(iv) For any proposition A which is not in $\{E_i\}$ or a disjunction of member of $\{E_i\}$, if the agent's degree of belief in A changes, the change in belief in A is not both caused by the environment's effects on the agent's sense organs and also not the result of conscious reasoning of any kind (i.e., the condition if (ii) and (iii) do not both hold for A).

and

(v) Conditions are such that at the end of the event the agent's degrees of belief in the members of $\{E_i\}$ are all reasonable.

If a partition $\{E_i\}$ satisfies the conditions (i)–(v), $\{E_i\}$ will be called the (generalized) observation's *observation set*.

The definition of G-observation's does away with strict observation's requirement of observational knowledge, so that we may now say without reservation that many, perhaps most of the events we commonly refer to as observations seem to constitute G-observations. All strict observations with non-conjunctive observed propositions are G-observations. If a strict observation's observed proposition is the conjunctive proposition

$A = \wedge_{i \in I} A_i$ it will either constitute a G-observation taking the observation set to be $\{A, \bar{A}\}$; or else the strict observation can be redescribed as a sequence of G-observations, with observation sets $\{A_i, \bar{A}_i\}$ occurring in sequence or simultaneously. If, as I have suggested, what are commonly taken to be strict observations are merely approximations to strict observations because the agent achieves strong reasonable belief but not the certainly required of knowledge, such observations are nonetheless correctly described as G-observations or sequences of G-observations. Finally, the kind of clear cut departures from strict observations described in Section 2.1. constitute G-observations. This kind of case may occur more frequently than we commonly suppose. Often we look or listen, our sense organs are effected by objects of perception, and we are caused to shift our beliefs without conscious inference and without arriving at new beliefs which approach anything like certainty. Such new beliefs are often reasonable because the agent's perceptual capacities, both innate and learned, are reliable.

At the same time, it must be born in mind that not all events which satisfy conditions (i)–(iv) in the definition of G-observation will qualify as G-observations. A crack on the head might put a man's mind in a deranged state in which the visual pattern presented to him when he looks at an ordinary tree will cause him to believe, without inference, that money grows on trees. This new belief will not qualify as reasonable, and the event will not qualify either as a strict observation that money grows on trees or as a generalized observation giving a reasonable and high degree of belief that money grows on trees.

The reader should be dissatisfied with my definitions of observation because they appeal to conditions for which no analysis is at hand. Not just any causally necessary condition nor just any part of a causally sufficient condition for a change of belief will count as the, or a, cause of a given change of belief. More obscure yet are the conditions under which a new belief caused by the environment's effects on the agent's sense organs will count as a reasonable belief. But the need to put such matters straight is an independently existing problem in the analysis of observation which need not detain us here. I take it to be a simple fact that many cases of observation satisfy the conditions of G-observations. If this is correct, we may appeal to our characterization of observation in explaining how information obtained by observation is to be brought to bear on our overall set of belief. And as long as we do not rely on any unstated assumptions about the conditions used in the definition, we may do so however these conditions are to be further analyzed.

2.5. *G-Observation and the Conditions for Generalized Conditionalization*

I turn now to applying the definition of *G*-observation in the task of determining when conditions (c) and (d) of principle *PG* hold for a partition $\{E_i\}$. 'Stability', 'new belief' and 'new reason' are defined as in Section 1.7. We shall say that a proposition *X stems from* the set $\{Y_j\}_{j \in J}$ just in case it is a member of $\{Y_j\}_{j \in J}$ or it is a disjunctions of members of $\{Y_j\}_{j \in J}$. The argument proceeds much as it did in Section 1.7., with the relation of stemming from here playing the same role formerly played by the relation of being a particle of.

Suppose that a *G*-observation takes place with observation set $\{E_i\}$. We want to argue that conditions (c) and (d) of principle *PG* hold for $\{E_i\}$. Conditions (i) and (v) in the definition of *G*-observations explicitly say that (c) holds for $\{E_i\}$. Turning to (d), clearly (d) holds only if the agent's beliefs are initially stable. So I will now argue that if a *G*-observation has occurred and the agent's beliefs are initially stable, condition (d) holds for $\{E_i\}$, the observation set of the *G*-observation.

Given the definition of 'stems from', above, the definition of 'indirectly rests on' given in 2.3., and the assumption of Section 2.3. that the reasons we are concerned with may be taken to be beliefs, condition (d) can be restated as:

(2.5.1.) After the change of belief any reasons the agent might have which in fact make reasonable or justify changes in belief in any proposition $A \notin \{E_i\}$ are either

(i) beliefs whose objects are propositions which stem from $\{E_i\}$,

or

(ii) beliefs at which the agent has arrived by a chain of reasoning whose original premises are beliefs whose objects are propositions which stem from $\{E_i\}$.

Since the agent's initial beliefs are assumed to be stable, any reason the agent might have which in fact makes reasonable or justifies changes in other beliefs must be a new reason. Let $P_n(R)$ be any such new reason. To prove (2.5.1.) we have to prove that

(2.5.2.) Either (i) R stems from $\{E_i\}$,
or (ii) The agent has arrived at the belief $P_n(R)$ by a chain of reasoning whose original premises are beliefs whose objects are propositions which stem from $\{E_i\}$.

I will prove (2.5.2.) by proving

(2.5.3.). If not (2.5.2. (i)), then (2.5.2. (ii)).

Assume that not (2.5.2. (i)), i.e. that R does not stem from $\{E_i\}$. $P_n(R)$ is either consciously reasoned or not consciously reasoned. Let us consider unreasoned $P_n(R)$ first. If $P_n(R)$ is also uncaused it is not reasonable, and so, as noted in Section 1.7., not after all, a new reason. If $P_n(R)$ is caused as well as unreasoned, it follows from (iv) in the definition of G-observation that it stems from $\{E_i\}$ after all. Since we are supposing that R does not stem from $\{E_i\}$, $P_n(R)$ must be reasoned. Since the agent's beliefs are assumed to be initially stable, $P_n(R)$ must be reasoned on the basis of new reasons $\{P_n(R_j)\}_{j \in J}$. If a $P_n(R_j)$ is unreasoned, as before it must be caused and so it must stem from $\{E_i\}$. If a $P_n(R_j)$ is reasoned the argument reapplies as it did in the case of the reasoned $P_n(R)$. Assuming that chains of reasoning are finite, such reapplications of the argument must come to an end in a case in which the basis of reasoning contains only beliefs whose objects are propositions which stem from $\{E_i\}$. Hence *all* the original premises which form the basis of reasoning for $P_n(R)$ are beliefs whose objects are propositions which stem from $\{E_i\}$, used at one or another stage of reasoning in support of $P_n(R)$. This completes the proof of (2.5.3.), and so of (2.5.2.), which was to be shown.

2.6. *Final Remarks*

I have shown in this paper that change of belief by conditionalization, both strict and generalized, can be justified in terms of qualitative principles of inductive reasoning, and that the difficult conditions of application of these principles can be ascertained by application of a corresponding analysis of observation. The conditions of application of the principles might also be taken to suggest, at least provisionally, limits on the correct application of conditionalization in describing reasonable change of belief. But the critic may charge that the arguments presented here accomplish no more than shifting the problem of justifying conditionalization to the problems of justifying principles P and PG and of further clarifying the analysis of strict and G-observation. One may agree with this criticism, and still maintain that we have made progress toward a complete account of reasonable change of belief. If not universally correct, principles P and PG must be agreed to hold in many circumstances. It seems clear that many events constitute G-observations; and, if we ever really come to have observational *knowledge*, many events constitute strict observations. And insofar as we have succeeded in reducing the problem of justifying conditionalization to problems about the analysis of observation and the justification of plausible qualitative principles of inductive reasoning, we have moved some way toward solving outstanding difficulties in epistemology.

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NOTES

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[1] Lewis reports that the argument was suggested to him by remark's of Hilary Putnam (p. 113, in [7], reprinted from a Voice of America Forum Lecture). Others have examined the possibility of carrying out essentially the same idea but wrongly concluded (e.g., Hacking [5], p. 315; also this author) that it could not be done.

[2] These interconnections, the argument for conditionalization, and the difficulty with it will all be detailed in Teller [12].

[3] My original proof was a slightly weaker form of the second one presented here. Arthur Fine discovered that the mathematically illuminating way to regard the proof was as indicated in the outline of the first proof. His version of the proof led to strengthening and shortening of my original formulation, resulting in the second proof below.

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